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Self-consistent effective-medium theory of random internal waves

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In the first part of this paper we introduce a path-integral formalism for the internal-wave field of the ocean. The intent is to show that this type of formalism may be useful in suggesting improvements to current calculations, as it provides a framework for applying a wide variety of approximations that have been and are currently being developed in other areas of physics. We demonstrate the method by deriving equations for a self-consistent field approach (also known as the direct-interaction approximation). The experience in other areas of physics is that the self-consistent field approximation is more reliable than lowest-order perturbation theory. The end result of the DIA is the determination of an effective linear model for the description of internal waves in the deep-ocean environment. In the second part of the paper we obtain Hasselmann's source function by a prescribed limiting process and are able to indicate possible improvements in related calculations by comparing the limiting assumptions with numerically computed values.

1. Introduction

In the past two decades the study of the transfer of energy in the internal-wave field via three-wave (nonlinear) interactions has become an area of active research. The possibility of three-wave *resonant* interactions was first mentioned by Phillips (1977). Subsequently a Boltzmann-like equation giving energy transfer rates in the Resonant-Interaction Approximation (RIA) was obtained by Hasselmann (1966, 1967) using multiple-time-scale methods. Inherent in his calculation was the assumption that transfer times were long compared with typical oscillation periods, a reasonable assumption given the widespread belief that internal-wave interactions were weak. Later synthesis of an analytic form for the internal-wave spectrum by Garrett & Munk (1972, 1975) raised hopes that results of numerical calculations using Hasselmann's transfer equation might contain some measure of reality. Unfortunately, subsequent calculations by Olbers (1976), McComas (1977) and Pomphrey, Meiss & Watson (1980) predicted very short interaction times in a rather large region of the spectrum, so that the results were internally inconsistent. What was needed, then, was a systematic way to relax the strict assumptions inherent in the derivation of Hasselmann. The primary purpose of this paper is to describe a formalism that might very well prove useful to that end.

In recent years new methods have been developed for analysing classical random nonlinear systems which partially owe their existence to analogous methods developed earlier for quantum many-body physics and particle physics. In the classical regime

most of the emphasis has been on critical dynamics and fully developed turbulence. We propose to formulate the dynamics of both internal waves and surface waves using these methods.

We should emphasize at the outset that we make no claim that the formalism developed herein provides a complete picture of the very complicated dynamics of internal waves. Rather, our interest is in demonstrating that the formalism provides a much more general framework in which to study internal waves than do the earlier schemes. In fact the reason for mentioning surface waves at all is to demonstrate that the formalism is easily generalized from that obtained for the simple three-wave interacting system used to model internal waves. It is in part this flexibility that makes this method powerful.

The aim of our program will be to calculate the linear response functions (Green's functions) and the two-time correlation functions. The ocean will be assumed to be a stationary random system. The time-development of perturbations to the wave field is described by the linear response functions.

In the theory of turbulence there is a long history of attempts at setting up such a formalism. The early work of Kraichnan (1959) and Wyld (1961) was followed by a 1973 paper by Martin, Siggia & Rose (1973) (henceforth MSR) which introduced the idea of doubling the number of variables in the problem, leading to the current formulation. There have been a number of papers which have rewritten the MSR formalism in various ways, see for example Phythian (1977) and Langouche, Roekaerts & Tirapegui (1979). We choose to use the path-integral formalism of Phythian to set up the problem. The reason for this is that there is currently a great deal of activity in various areas of physics using path integrals. This will enable us easily to make use of some of their techniques. In this paper we will discuss perturbative and self-consistent evaluations of the path integral. In fact for the approximation we propose, we could use directly the original work of Kraichnan (1959) who proposed the direct-interaction approximation which has been extensively studied. We refer the reader to the reviews of Leslie (1973) and Orzag (1977). For a similar approximation see Phythian (1969), Herring (1965) and Edwards (1964). It is essentially a self-consistent field approximation, and, in the language of diagrams, vertex corrections are ignored. The semi-classical or WKB approximation is another standard approximation to path integrals. We will not discuss that here, but it does appear that it is the appropriate approximation for part of the internal-wave field.

In order to show how such a formalism might shed more light on previous calculations we show that a particular limiting case of our model precisely corresponds to the results of Hasselmann and consequently indicates how one might improve on the calculations of Olbers and McComas. We also show that the formalism splits Hasselmann's source function into two pieces, which have natural interpretations as effective damping and driving forces. This finally leads us to interpret our model of the internal-wave field as a collection of uncoupled, damped oscillators in the presence of random driving forces, in the spirit of effective-medium theories found in other areas of physics.

In § 2 we review the derivation of the path integral for stationary random processes. In § 3 we discuss the correlation functions and the linear-response functions and the direct-interaction approximation. In § 4 we interpret the nonlinear problem as an effective linear theory with a memory. In § 5 we discuss the relationship of our model to the results of Hasselmann. In § 6 we briefly mention the WKB approximation and

the important physics that is involved. The final section is devoted to a summary and discussion of some unresolved problems.

2. Path integral

In this section we will sketch the derivation of the path-integral form of the characteristic functional describing the random field. For some of the more technical mathematical points we refer to the literature, Phythian (1977) and Langouche *et al.* (1979).

As far as is possible we will use the notation of Olbers (1976) for the mode description. The equations to be solved are of the form

$$\dot{A}^s(\mathbf{k},t) + i\Omega^s(\mathbf{k})\,A^s(\mathbf{k},t) + \nu_0(\mathbf{k})\,A^s(\mathbf{k},t) + \sum_{s_1,s_2}\int d^3\mathbf{k}_1\,d^3\mathbf{k}_2\,\delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,B^{-s\,s_1\,s_2}_{-k\,k_1\,k_2}A^{s_1}(\mathbf{k}_1,t)\,A^{s_2}(\mathbf{k}_2,t) = f^s(\mathbf{k},t), \quad s = \pm 1. \tag{2.1}$$

Here we have introduced the terms $\nu_0(\mathbf{k})\,A^s(\mathbf{k},t)$ and $f^s(\mathbf{k},t)$ as external damping and driving forces on the internal-wave field. The motivation for the particular choice of these terms is twofold. The first is that of simplicity – we wish to demonstrate the application of the formalism to this problem without introducing complexities that might tend to obscure the general ideas. The second motivation can only be seen *a posteriori*. When we subsequently write down the Dyson's equations for this problem it will be seen that new terms appear which have the same formal structure as the two 'external' terms; this leads us to interpret the new terms as 'corrections' to the initial terms, i.e. as 'effective' damping and driving terms. For most modes f and ν_0 are negligible and can be ignored. They are just a convenient way of parametrizing the coupling of inertial waves to the rest of the world.

The amplitudes, mode frequencies and couplings satisfy the relations:

$$A^s(\mathbf{k},t) = [A^{-s}(-\mathbf{k},t)]^*, \quad \Omega^s(\mathbf{k}) = \Omega^{-s}(-\mathbf{k}), \tag{2.2), (2.3}$$

$$B^{s_1 s_2 s_3}_{k_1 k_2 k_3} = (B^{-s_1-s_2-s_3}_{-k_1-k_2-k_3})^*. \tag{2.4}$$

The index s is either $+n$ or $-n$ with $\Omega^+ > 0$. The label n is an additional mode index, which for internal waves is equal to unity. If an Ω appears without the label, it is taken to be Ω^+. We make the *ad hoc* assumption that the driving forces $f^s_0(\mathbf{k},t)$ are stationary and random with a Gaussian distribution function,

$$\langle f^s(\mathbf{k},t) f^{s'}(\mathbf{k}',t')\rangle = \delta^3(\mathbf{k}+\mathbf{k}')\,\delta_{s,-s'}\,R_0(\mathbf{k},t-t'). \tag{2.5}$$

This particular assumption allows us to obtain closed-form solutions for the path integrals that will appear later. Other models for the interaction of the external forces with the internal waves could be used, such as random parametric couplings in (2.1), but such models are somewhat more difficult to treat.

The probability distributions for f are specified in terms of its characteristic functional $C(\phi)$,

$$C(\phi) = \left\langle \exp\left\{ i\int d^3\mathbf{k}\sum_s \int_{-\infty}^{\infty} dt\,\phi^{-s}(-\mathbf{k},t) f^s(\mathbf{k},t)\right\}\right\rangle, \tag{2.6a}$$

$$C(\phi) = \exp\{-\tfrac{1}{2}\sum_s \int d^3\mathbf{k}\int dt\,dt'\,\phi^{-s}(-\mathbf{k},t)\,\phi^s(\mathbf{k},t')\,R_0(\mathbf{k},t-t')\}. \tag{2.6b}$$

The equations are much simpler when expressed in terms of their Fourier transforms,

$$\left.\begin{aligned} A^s(\mathbf{k},\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} A^s(\mathbf{k},t)\,dt, \\ \phi^s(\mathbf{k},\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \phi^s(\mathbf{k},t)\,dt, \\ F_0(\mathbf{k},\omega) &= \int_{-\infty}^{\infty} e^{i\omega t} R(\mathbf{k},t)\,dt. \end{aligned}\right\} \tag{2.7}$$

With this convention, equation (2.6) becomes

$$C(\phi) = \exp\left\{-\frac{2\pi}{2}\sum_s \int d^3\mathbf{k} \int d\omega\, \phi^{-s}(-\mathbf{k},-\omega)\, F_0(\mathbf{k},\omega)\, \phi^s(\mathbf{k},\omega)\right\}. \tag{2.8}$$

We are interested in finding the probability distribution for the amplitudes A. To that end we introduce the characteristic functional for those variables,

$$Z(\xi) = \langle \exp i\sum_s \int d^3\mathbf{k} \int dt\, \xi^{-s}(-\mathbf{k},t)\, A^s(\mathbf{k},t)\rangle$$

or in terms of the Fourier-transformed amplitudes

$$Z(\xi) = \langle \exp\left[2\pi i \sum_s \int d^3\mathbf{k} \int d\omega\, \xi^{-s}(-\mathbf{k},-\omega)\, A^s(\mathbf{k},\omega)\right]\rangle. \tag{2.9}$$

The angle brackets refer to averaging over the distribution for f. They are given formally as

$$\langle X(f)\rangle = \int \mathscr{D}f\, W(f)\, X(f) \tag{2.10}$$

where $W(f)$ is the probability distribution function for f and is just the Fourier transform of C. The integral is a functional integral in that we are to integrate over all possible functions $f(\mathbf{k},t)$ for each mode k. One way to achieve this is to discretize $\mathbf{k}$ and t and integrate,

$$\mathscr{D}f \equiv \prod_k \prod_n df(\mathbf{k},t_n). \tag{2.11}$$

For the purposes of this paper it is not necessary to be specific and Phythian (1977) can be consulted for more details. The idea now is to replace $W(f)$ by its Fourier transform $C(\phi)$ and to change variables from f to A using (2.1):

$$\langle X\rangle = \int X \frac{\mathscr{D}f}{\mathscr{D}A}\, \mathscr{D}A\, \mathscr{D}\phi\, C(\phi) \exp\{-i\sum_s \int d^3\mathbf{k} \int dt\, \phi^{-s}(-\mathbf{k},t) f^s(\mathbf{k},t)\}, \tag{2.12}$$

where it is understood that f is to be replaced by

$$\begin{aligned} f_s(\mathbf{k},\omega) = {} & \{-i\omega + i\Omega^s(\mathbf{k}) + \nu_0(\mathbf{k})\} A^s(\mathbf{k},\omega) \\ & + \sum_{s_1}\sum_{s_2} \int d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, \delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, \delta(\omega-\omega_1-\omega_2) \\ & \times A^{s_1}(\mathbf{k}_1,\omega_1)\, A^{s_2}(\mathbf{k}_2,\omega_2)\, B^{-ss_1s_2}_{-kk_1k_2}. \end{aligned} \tag{2.13}$$

The driving function f has disappeared from the problem and its characteristic functional $C(\phi)$ has appeared. There are now two functional integrals, $\mathscr{D}A$ and $\mathscr{D}\phi$ and the Jacobian $\mathscr{D}f/\mathscr{D}A$ in place of $\mathscr{D}f$. The Jacobian can be evaluated (Phythian

1977; Langouche *et al.* 1979) and is given formally by

$$\frac{\mathscr{D}f}{\mathscr{D}A} = \exp\left\{+\int_{-\infty}^{\infty} d\omega \sum_{s,s_1} \int d^3\mathbf{k}\, B^{s-ss_0}_{k-k0} A^{s_1}(0,0)\right\}. \tag{2.14}$$

It has been shown (Langouche *et al.* 1979; Dominicis & Peliti 1978) that, if diagrammatic perturbation theory is used, the Jacobian cancels 'tadpole' diagrams and both the tadpoles and the Jacobian can be omitted. See the appendix for a discussion of this point.

We now extend the definition of the angle brackets to include ϕ functionals as well as $A^s(\mathbf{k},\omega)$ functionals,

$$\langle X(A,\phi)\rangle = \int \mathscr{D}A\,\mathscr{D}\phi \frac{\mathscr{D}f}{\mathscr{D}A} X(A,\phi)\, C(\phi) \times \exp\{-2\pi i \sum_s \int d^3k\, d\omega\, \phi^{-s}(-\mathbf{k},-\omega) f^s(\mathbf{k},\omega)\}. \tag{2.15}$$

The generating functional introduced in (2.9) is now extended to include ϕ,

$$Z(\xi,\eta) = \langle \exp\{2\pi i \sum_s \int d^3\mathbf{k} \int d\omega\, [\xi^{-s}(-\mathbf{k},-\omega) A^s(\mathbf{k},\omega) + \eta^{-s}(-\mathbf{k},-\omega)\phi^s(\mathbf{k},\omega)]\}\rangle. \tag{2.16}$$

As we are presumably interested in correlation functions of $A^s(\mathbf{k},\omega)$ which can be obtained by differentiation with respect to ξ, it seems unnecessary to introduce η. However, the field ϕ (and hence η) plays a crucial role in defining a response function and in simplifying the perturbation series. To demonstrate this we imagine replacing $f \to f + e$ in the equation of motion. This shows up in (2.15) as an additional factor in the integral

$$\exp\{+2\pi i \sum_s \int d^3\mathbf{k} \int d\omega\, \phi^{-s}(-\mathbf{k},-\omega)\, e^s(\mathbf{k},\omega)\}.$$

If we expand this to first order in e and consider the change in $\langle A\rangle$ we see that

$$\left\langle \frac{\delta A^s(\mathbf{k},\omega)}{\delta e^{-s'}(-\mathbf{k}',-\omega')} \right\rangle = 2\pi i \langle A^s(\mathbf{k},\omega)\, \phi^{s'}(\mathbf{k}',\omega)\rangle, \tag{2.17}$$

which can also be written

$$\left\langle \frac{\delta A^s(\mathbf{k},t)}{\delta e^{-s'}(-\mathbf{k}',t')} \right\rangle = i\langle A^s(\mathbf{k},t)\, \phi^{s'}(\mathbf{k}',t')\rangle. \tag{2.18}$$

Thus the term on the right-hand side of (2.18) gives the linear response of $A^s(\mathbf{k},t)$ to a known perturbation. It is linear only in the sense that e is treated to first order. The rest of the dynamics is nonlinear. Response and correlation functions are calculated by taking the appropriate functional derivatives of Z, for instance

$$\langle A^s(\mathbf{k},\omega)\, \phi^{s'}(\mathbf{k}',\omega')\rangle = \frac{1}{Z}\left(\frac{1}{2\pi i}\right)^2 \frac{\delta}{\delta\xi^{-s}(-\mathbf{k},-\omega)} \frac{\delta}{\delta\eta^{-s'}(-\mathbf{k}',-\omega')} Z(\xi,\eta)\Big|_{\xi=\eta=0}. \tag{2.19}$$

The standard trick used to develop perturbation theory is to notice that, if the theory were linear, all of the integrals in (2.16) could be obtained exactly. In the actual case the factor preventing exact analytical integration is given by

$$\exp\{-2\pi i \sum_{s,s_1,s_2} \int d^3\mathbf{k}\, d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, \delta(\omega-\omega_1-\omega_2)\, \delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \times B^{-ss_1s_2}_{-kk_1k_2} A^{s_1}(\mathbf{k}_1,\omega_1)\, A^{s_2}(\mathbf{k}_2,\omega_2)\, \phi^{-s}(-\mathbf{k},-\omega)\}.$$

If we recognize that $A^s(\mathbf{k},\omega)$ can be replaced by

$$A^s(\mathbf{k},\omega)\to\frac{1}{2\pi i}\frac{\delta}{\delta\xi^{-s}(-\mathbf{k},-\omega)}$$

and each ϕ by

$$\phi^s(\mathbf{k},\omega)\to\frac{1}{2\pi i}\frac{\delta}{\delta\eta^{-s}(-\mathbf{k},-\omega)},$$

then

$$Z(\xi,\eta)=\exp\left\{\left(\frac{1}{2\pi}\right)^2\int\delta(\omega_1+\omega_2+\omega_3)\,\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,d\omega_1\,d\omega_2\,d\omega_3\,d^3\mathbf{k}_1\,d^3\mathbf{k}_2\,d^3\mathbf{k}_3\right.$$
$$\left.\times\sum_{s,s_1,s_2}B^{s_1s_2s_3}_{k_1k_2k_3}\frac{\delta}{\delta\eta^{-s_1}(-\mathbf{k}_1,-\omega_1)}\frac{\delta}{\delta\xi^{-s_2}(-\mathbf{k}_2,-\omega_2)}\frac{\delta}{\delta\xi^{-s_3}(-\mathbf{k}_3,-\omega_3)}\right\}Z_0(\xi,\eta). \tag{2.20}$$

The term $Z_0(\xi,\eta)$ is obtained from (2.15) and (2.16) by finding all of the integrals now that the nonlinear piece has been removed. The result is

$$Z_0(\xi,\eta)=\exp-\left\{\frac{2\pi}{2}\int d\omega\int d^3\mathbf{k}\sum_s\xi^{-s}(-\mathbf{k},-\omega)\frac{F_0(\mathbf{k},\omega)}{D^{-s}(-\mathbf{k},-\omega)\,D^s(\mathbf{k},\omega)}\xi^s(\mathbf{k},\omega)\right\}$$
$$\times\exp\left\{2\pi i\int d\omega\int d^3\mathbf{k}\sum_s\xi^{-s}(-\mathbf{k},-\omega)\frac{1}{D^s(\mathbf{k},\omega)}\eta_s(\mathbf{k},\omega)\right\}. \tag{2.21}$$

The new functions D are

$$D^s(\mathbf{k},\omega)=\nu_0(\mathbf{k})+i\Omega^s(\mathbf{k})-i\omega,\quad D^+(\mathbf{k},\omega)=[D^-(-\mathbf{k},-\omega)]^*. \tag{2.22}$$

We can now write down a diagram expansion for the perturbation series. We do that in the next section.

In principle quantities of interest can be calculated directly from (2.15) without resorting to the perturbation theory that follows from (2.20). Currently there is a major effort aimed at extracting non-perturbative information directly from path integrals describing a wide variety of physical phenomena. As those methods tend to be difficult to implement, we propose to first test the SCFA (DIA). There are a number of quantities of interest for the internal-wave field. However, there is relatively meagre information on higher moments so in this paper we will be primarily interested in the two-point functions,

$$\left.\begin{aligned}\frac{1}{2\pi}U^{ss'}(\mathbf{k},\omega)\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')&=\langle A^s(\mathbf{k},\omega)\,A^{s'}(\mathbf{k}',\omega')\rangle,\\ \hat{U}^{ss'}(\mathbf{k},t)\,\delta^3(\mathbf{k}+\mathbf{k}')&=\langle A^s(\mathbf{k},t)\,A^{s'}(\mathbf{k},0)\rangle,\end{aligned}\right\} \tag{2.23}$$

and

$$\left.\begin{aligned}\frac{1}{2\pi}G^{ss'}(\mathbf{k},\omega)\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')&=i\langle A^s(\mathbf{k},\omega)\,\phi^{s'}(\mathbf{k}',\omega')\rangle,\\ \hat{G}^{ss'}(\mathbf{k},t)\,\delta^3(\mathbf{k}+\mathbf{k}')&=i\langle A^s(\mathbf{k},t)\,\phi^{s'}(\mathbf{k}',0)\rangle.\end{aligned}\right\} \tag{2.24}$$

The definitions of U and G lead to some important symmetry properties:

$$U^{ss'}(\mathbf{k},\omega)=[U^{-s-s'}(-\mathbf{k},-\omega)]^*,\quad U^{ss'}(\mathbf{k},\omega)=U^{s's}(-\mathbf{k},-\omega),$$
$$G^{ss'}(\mathbf{k},\omega)=[G^{-s-s'}(-\mathbf{k},-\omega)]^*.$$

3. Perturbation theory

In this section we will exhibit the general structure of the two-point functions and discuss briefly the diagrammatic perturbation theory. We begin by listing the free Green's functions and correlation functions, and we will then point out the general features that the exact functions will have. The free correlation function is defined by

$$\begin{aligned}\frac{1}{2\pi} U_0^s(\mathbf{k},\omega)\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\delta_{s,-s'} &= \langle A^s(\mathbf{k},\omega)\,A^{s'}(\mathbf{k}',\omega')\rangle_0 \\ &= \frac{1}{Z_0}\left(\frac{1}{2\pi i}\right)^3 \frac{\delta}{\delta\xi^{-s}(-\mathbf{k},-\omega)}\frac{\delta}{\delta\xi^{-s'}(-\mathbf{k}',-\omega')} Z_0\Big|_{\xi=\eta=0} \\ &= \frac{1}{2\pi}\frac{1}{D^{-s}(-\mathbf{k},-\omega)} F_0(\mathbf{k},\omega)\frac{1}{D^s(\mathbf{k},\omega)}\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\delta_{s,-s'}, \qquad (3.1)\end{aligned}$$

$$U_0^s(\mathbf{k},\omega) = \frac{F_0(\mathbf{k},\omega)}{\nu_0^2(\mathbf{k})+[\omega-\Omega^s(\mathbf{k})]^2}.$$

The free-response function is given by

$$\begin{aligned}\frac{1}{2\pi} G_0^s(\mathbf{k},\omega)\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\delta_{s,-s'} &= i\langle A^s(\mathbf{k},\omega)\,\phi^{s'}(\mathbf{k}',\omega')\rangle \\ &= i\left(\frac{1}{2\pi i}\right)^2 \frac{1}{Z_0}\frac{\delta}{\delta\xi^{-s}(-\mathbf{k},-\omega)}\frac{\delta}{\delta\eta^{-s'}(-\mathbf{k}',-\omega')}(Z_0)\Big|_{\xi=\eta=0} \\ &= \frac{1}{2\pi}\frac{1}{\nu_0(\mathbf{k})+i\Omega^s(\mathbf{k})-i\omega}\,\delta^3(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')\,\delta_{s,-s'}. \qquad (3.2)\end{aligned}$$

If F_0 is independent of ω, the corresponding time-dependent functions are

$$\hat{U}_0^s(\mathbf{k},t) = \frac{F_0(\mathbf{k})}{2\nu_0(\mathbf{k})}\,e^{i\Omega(\mathbf{k})t}\,[\theta(t)\,e^{-\nu_0 t}+\theta(-t)\,e^{\nu_0 t}] \qquad (3.3)$$

and

$$\hat{G}_0^s(\mathbf{k},t) = e^{-it\Omega^s(\mathbf{k})}\,e^{-\nu_0 t}\,\theta(t). \qquad (3.4)$$

The exact functions will have the same structure as that exhibited in (3.1) and (3.2), except that F_0, ν_0 and Ω will be replaced by new functions that depend on ω. To the extent that the ω dependence of the effective f, ν and Ω can be ignored, (3.3) and (3.4) are exact. The ω dependence will modify at least the short-time behaviour. If the singularity nearest the real ω axis remains a simple pole, the long-time behaviour will have the structure of (3.3) and (3.4). We now proceed to develop the perturbation scheme to calculate the effective f, ν and Ω using standard diagrammatic techniques. We list the rules for calculating diagrams:

(*a*) There are two kinds of propagators, one corresponding to correlation functions and the other to response functions. Each line carries an s, $\mathbf{k}$, and ω. The response functions have an arrow associated with them, and we will conventionally take the quantities s, $\mathbf{k}$ and ω to flow with the arrow. The lines and their corresponding functions are shown in figure 1.

(*b*) Each vertex has exactly one line with an arrow pointing toward the vertex and

$$\frac{i}{D^s(\mathbf{k},\omega)} = \quad s, \mathbf{k}, \omega \longrightarrow$$

$$\frac{-F(\mathbf{k})}{D^{-s}(-\mathbf{k},-\omega)\,D^s(\mathbf{k},\omega)} = \quad s, \mathbf{k}, \omega \longrightarrow$$

FIGURE 1. Lines representing response function and correlation function.

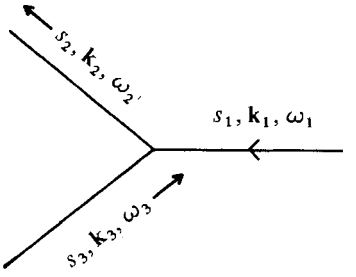

FIGURE 2. Lowest-order vertex function. The line with the incoming arrow is the first argument $B^{s_1-s_2s_3}_{k_1-k_2k_3} \times \delta^3(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3) \times \delta(\omega_1-\omega_2+\omega_3)$.

FIGURE 3. Lowest-order graph contributing to the correlation function, U. Because of its symmetry there is a factor of $\frac{1}{2}$ associated with it.

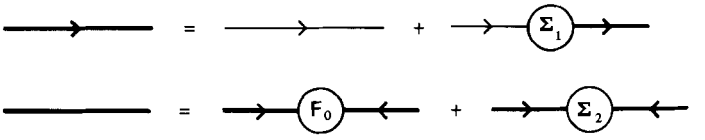

FIGURE 4. Dyson functions for the two-point functions.

two other lines that either have arrows pointing away from the vertex or no arrows. In figure 2 we show a vertex which has associated with it the factor

$$B^{s_1-s_2+s_3}_{k_1-k_2+k_3}\,\delta^3(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3)\,\delta(\omega_1-\omega_2+\omega_3).$$

The first pair corresponds to the arrow in the diagram. B is symmetric in the other pairs.

(*c*) All internal s, $\mathbf{k}$ and ω's are summed over.

(*d*) For Nth-order perturbation theory, draw all topologically distinct diagrams. Graphs possessing a symmetry have a symmetry factor; for example the graph of figure 3 gets a factor of $\frac{1}{2}$.

(*e*) Each diagram gets a factor $(2\pi)^{-\frac{1}{2}(N+Ne)}$.

(*f*) Each external A field gets a factor of $(-i)$. (The extra i here is because the response function is $i\langle A\phi\rangle$, see (3.2).)

We mention that, if vertices with more lines are present, the only rule that is changed is (*e*). Each vertex will still have only one line with an arrow pointing into it.

The Dyson equations have been given by MSR. For completeness we show them in figures 4–6. Note that, although there is only one kind of bare vertex, there are three kinds of renormalized vertices which are shown to lowest order in figure 6.

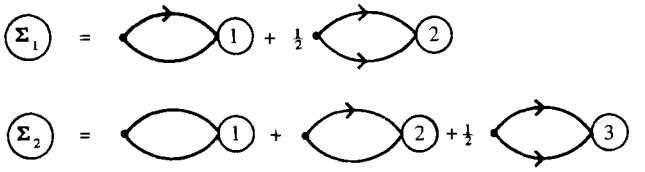

FIGURE 5. Equation for $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ in terms of renormalized vertices.

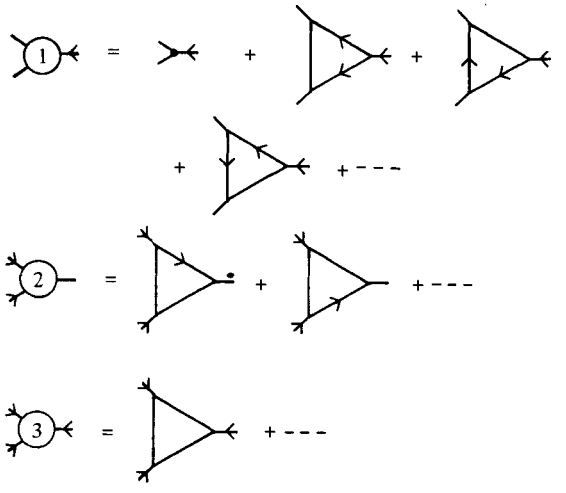

FIGURE 6. First terms in expansion of vertices.

Finally, it can easily be shown that any graph containing a closed loop whose lines are entirely response functions (arrows) is identically zero.

The Dyson equations are easily solved to give

$$G^{ss'}(\mathbf{k},\omega) = [D^{-s}(\mathbf{k},\omega)\,\delta_{s,-s'} - i\Sigma_1^{ss'}(\mathbf{k},\omega)]^{-1} \tag{3.5}$$

(note that this is a 2×2 matrix equation). The correlation function is given by

$$U^{ss'} = G^{ss_1}(\mathbf{k},\omega)\,[F_0(\mathbf{k},\omega)\,\delta_{s_1,-s_2} + \Sigma_2^{s_1 s_2}(\mathbf{k},\omega)]\,G^{s's_2}(-\mathbf{k},-\omega) \tag{3.6}$$

(implies summation over s_1, s_2).

If vertex corrections are ignored (direct interaction approximation) then we have

$$\Sigma_1^{ss'}(\mathbf{k},\omega) = \frac{2}{\pi}\sum_{\substack{s_1 s_1' \\ s_2 s_2'}}\int d^3\mathbf{k}_1\,d^3\mathbf{k}_2\,d\omega_1\,d\omega_2\,\delta^3(\mathbf{k}+\mathbf{k}_2-\mathbf{k}_1)\,\delta(\omega+\omega_2-\omega_1)$$
$$\times B_{kk_2-k_1}^{-ss_2s_1}\,B_{k_1-k_2-k}^{s_1's_2'-s'}[iG^{s_1s_1'}(\mathbf{k}_1,\omega_1)][-U^{s_2s_2'}(\mathbf{k}_2,\omega_2)] \tag{3.7}$$

and

$$\Sigma_2^{ss'}(\mathbf{k},\omega) = \frac{1}{\pi}\sum_{\substack{s_1 s_1' \\ s_2 s_2'}}\int d^3\mathbf{k}_1\,d^3\mathbf{k}_2\,d\omega_1\,d\omega_2\,\delta^3(\mathbf{k}+\mathbf{k}_2+\mathbf{k}_1)\,\delta(\omega+\omega_2+\omega_1)$$
$$\times B_{kk_1k_2}^{-ss_1s_2}\,B_{-k-k_1-k_2}^{-s's_1's_2'}\,U^{s_1s_1'}(\mathbf{k}_1,\omega_1)\,U^{s_2s_2'}(\mathbf{k}_2,\omega_2); \tag{3.8}$$

$\mathbf{\Sigma}_2$ has the same symmetry properties as $\mathbf{U}$ and $i\mathbf{\Sigma}_1$ has the same as $\mathbf{G}$.

One of the assumptions of weak-interaction theory is that the diagonal elements of $\mathbf{U}$ and $\mathbf{G}$ can be ignored. $\mathbf{U}$ and $\mathbf{G}$ are assumed to have the form $U^{ss'} = U^s\delta_{s,-s'}$. The corresponding statement for $\mathbf{\Sigma}$ is $\Sigma^s = \Sigma^{-ss}$.

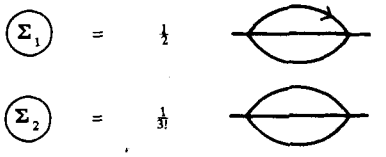

FIGURE 7. Graphs for $\mathbf{\Sigma}$ for four-wave interactions.

We now comment briefly on the application of these techniques to surface waves. The principal difference is that the equations involve four-wave coupling, that is there is a term

$$B \times A \times A \times A$$

in the equation of motion, (1.1). The term involving only two A's is still there, but it is not possible to have three-wave resonances. This means that in (3.12) the limit $\nu \to 0$ gives zero and the answer must come from finite ν which presumably gives a small contribution. In any case if we replace the BAA terms by $BAAA$ in (2.1) we obtain a very similar theory. The graphs for $\mathbf{\Sigma}$ are shown in figures 7 and they are to be solved self-consistently with the diagrams of figure 4. The number associated with a vertex is now 3! instead of 2!. There are some additional (2π) factors. The net effect is to change rule (e),

$$(2!)^N (2\pi)^{-\frac{1}{2}(N+Ne)} \to (3!)^N (2\pi)^{-\frac{1}{2}Ne+N},$$

and rule (b) gets a factor of i for each vertex. These two changes can be stated in a way that is easily generalized. If there are N_A A fields at a vertex, the vertex has associated with it $(i^{N_A+1})N_A!$. Each closed loop has associated with it an integral $\int d\omega/2\pi$.

4. Linear effective medium

In this section we shall elaborate on the effective medium picture and discuss the direct-interaction approximation and the relationship to the theory due to Hasselmann. We will first remind the reader of a few simple properties of linear systems, and compare these with our effective medium. The simplest situation assumes no memory and is given by

$$\dot{A}^s + i\Omega^s A^s + \nu A^s = f^s(t), \tag{4.1}$$

which is (2.1) with $B = 0$ and the mode index k ignored. If we define

$$\hat{U}(t, t') = \langle A^*(t) A(t') \rangle, \tag{4.2}$$

then in the steady state U depends only on $t - t'$ and is given by

$$\hat{U}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{F e^{i\omega t}}{(\nu - i\Omega + i\omega)(\nu + i\Omega - i\omega)}. \tag{4.3}$$

If we calculate $d\hat{U}(t,t)/dt$ we obtain

$$d\hat{U}(t,t)/dt = -2\nu\hat{U}(0) + F = 0, \tag{4.4}$$

where F is given by (2.5) and (2.7). This just represents the balance between the driving forces and the dissipative forces. We now compare with the correlation function obtained in the previous section,

$$\hat{U}^s(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega t} \left[\frac{1}{\nu_0 - i\Omega + i\omega - i\Sigma_1^-(-\omega)}\right] (F_0 + \Sigma_2^s) \left[\frac{1}{\nu_0 + i\Omega - i\omega - i\Sigma_1^+(\omega)}\right]. \tag{4.5}$$

We have suppressed the mode index k. If Σ_1, Σ_2 are approximately independent of ω, then it is clear that real Σ_1 is a frequency shift and $\mathscr{I}\Sigma_1$ is an effective viscosity representing loss to the other modes. This particular interpretation may be somewhat misleading, however. Consider the case where the mode in question is Doppler-shifted because of passage through an inhomogeneous medium (large-scale wave). In this case we would formally say that one mode has been damped while another mode of neighbouring wavenumber has grown. In reality of course we are just seeing the original wave with a slightly different frequency.

In addition to the 'effective damping' term there is also an effective driving force due to the other modes and its correlation function is Σ_2. The correlation function for the total driving force is thus $\Sigma_2 + F_0$.

Now in general Σ_1 and Σ_2 will depend on ω. In this case it is easy to see that the equation giving the evolution of the amplitudes is given by

$$\dot{A}(t) + \int_{-\infty}^{\infty} \hat{\Gamma}(t-t')\,A(t')\,dt' = f(t), \quad \hat{\Gamma}(t) = 0, \quad t < 0. \tag{4.6}$$

If we write

$$\Gamma^{+}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{\Gamma}(t)\,dt, \quad \Gamma^{-}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\hat{\Gamma}^{*}(t), \tag{4.7}$$

then we see that $\Gamma^{\pm}(\omega)$ is analytic in ω for $\mathscr{I}\omega > 0$. The corresponding response functions are

$$G^{\pm}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega t}}{\Gamma^{\pm}(\omega) - i\omega}, \tag{4.8}$$

which leads to an obvious interpretation of $\Sigma_1^{\pm}(\omega)$,

$$\Gamma^{+}(\omega) = \nu_0 + i\Omega - \Sigma_1^{+}(\omega). \tag{4.9}$$

The fact that $G(t) = 0$ for $t < 0$ (causality) implies that there are no zeros of $\Gamma(\omega) - i\omega$ in the upper half-plane. Because of these analytical properties of G and Γ, Kramer's Krönig dispersion relations can be used to calculate the real part of G and Σ once the imaginary part is known (or vice versa).

The equilibrium condition relating the energy, dissipation and driving forces is obtained by evaluating

$$U(t) = \int_{-\infty}^{+\infty} e^{-i\omega t}\frac{F(\omega)}{2\pi}\frac{d\omega}{[\Gamma^{-}(-\omega) - i\omega][\Gamma^{+}(\omega) - i\omega]}$$

at $t = 0$.

5. Relationship to previous results

In this section we obtain the results of Hasselmann as a special case of our model equation. The limiting assumptions made in obtaining the results provide a more transparent way to check the validity of the assumptions inherent in Hasselmann's equation; in the end of this section we indicate where improvements could be made in his scheme.

To obtain the results of Hasselmann one performs the following operations on (3.7) and (3.8) (using only $\Sigma^{s} \equiv \Sigma^{-ss}$).

(1) Assume that F_0, Σ_1, and Σ_2 are independent of ω and perform the ω_1 and ω_2 integrations. For example, defining $\nu(\mathbf{k}) \equiv \nu_0(\mathbf{k}) + \mathscr{I}\Sigma_1(\mathbf{k})$, we get

$$\int d\omega_1 d\omega_2 \delta(\omega-\omega_1-\omega_2)\,[\nu(\mathbf{k}_1)^2+(\Omega(\mathbf{k}_1)-\omega_1)^2]^{-1}\,[\nu(\mathbf{k}_2)+i\Omega(\mathbf{k}_2)-i\omega_2]^{-1}$$
$$= \frac{\pi i}{\nu(\mathbf{k}_1)}\,[\omega-\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_2)-i(\nu(\mathbf{k}_1)+\nu(\mathbf{k}_2))]$$
$$\times\,[(\omega-\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_2))^2+(\nu(\mathbf{k}_1)+\nu(\mathbf{k}_2))^2]^{-1}. \quad (5.1)$$

(2) Take the limit F_0, ν_0, Σ_1, $\Sigma_2 \to 0$ (in such a way that $\hat{U}(\mathbf{k}, 0) = F/2\nu$ is maintained). Then (5.1) becomes

$$\frac{\pi^2}{\nu(\mathbf{k}_1)}\,\delta(\omega-\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_2))+\frac{i\pi}{\nu(\mathbf{k}_1)}\,\frac{\mathscr{P}}{\omega-\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_2)}, \quad (5.2)$$

where $\mathscr{P}$ represents a principal part integral. When the actual structure of the B coefficients is considered it is easily seen that the delta function contributes to $\nu(k)$ and the principal part to a frequency shift. The Dirac delta function has come from the limit

$$\frac{\nu}{\nu^2+(\Delta\Omega)^2} \xrightarrow[\nu\to 0]{} \pi\delta(\Delta\Omega) \quad (5.3)$$

and we call the function on the left-hand side the frequency filter for the problem. Such a function was earlier obtained by Holloway & Henderson (1977) and Holloway (1978, 1979), where he also noted the RIA as a limiting case. This is the first indication of a possible improvement in the calculations of Olbers and McComas. What they did, in essence, was to assume that ν was small compared with typical oscillation frequencies and then calculated values of ν on the order of or even greater than the frequencies. The earlier assumption of neglecting the diagonal elements of $\mathbf{U}$, $\mathbf{G}$ corresponds approximately to assuming $\nu/\Omega \ll 1$.

Equations (3.7) and (3.8) must be solved self-consistently, however, so that values of $\nu(\mathbf{k})$ used for input on the right-hand side are the same as those obtained on the left-hand side. Unfortunately solving these equations self-consistently does involve integration over one more variable, and so may require an inordinate amount of computer time (the two-dimensional integral is already relatively time-consuming).

Having now carried out the limiting process, we are ready to make a detailed comparison with the source function of Hasselmann, which he writes as

$$\mathscr{S} = \int d^3\mathbf{k}_1\, d^3\mathbf{k}_2\,\{T^+\delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\Omega-\Omega_1-\Omega_2)\,[n_1n_2-nn_1+nn_2]$$
$$+\,2T^-\delta^3(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)\,\delta(\Omega-\Omega_1+\Omega_2)\,[n_1n_2+nn_1-nn_2], \quad (5.4)$$

where

$$n(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}') = \langle A^+(\mathbf{k}, t)\,A^-(\mathbf{k}', t)\rangle\,\frac{1}{\Omega(\mathbf{k})}.$$

We split $\mathscr{S}$ into two pieces,

$$\mathscr{S}_2(\mathbf{k}) = \int d^3\mathbf{k}_1\, d^3\mathbf{k}_2\,\{T^+\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\Omega-\Omega_1-\Omega_2)\,n_1n_2$$
$$+\,2T^-\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2)\,\delta(\Omega-\Omega_1+\Omega_2)\,n_1n_2\} \quad (5.5)$$

and

$$\mathscr{S} = \mathscr{S}_2 - \mathscr{S}_1.$$

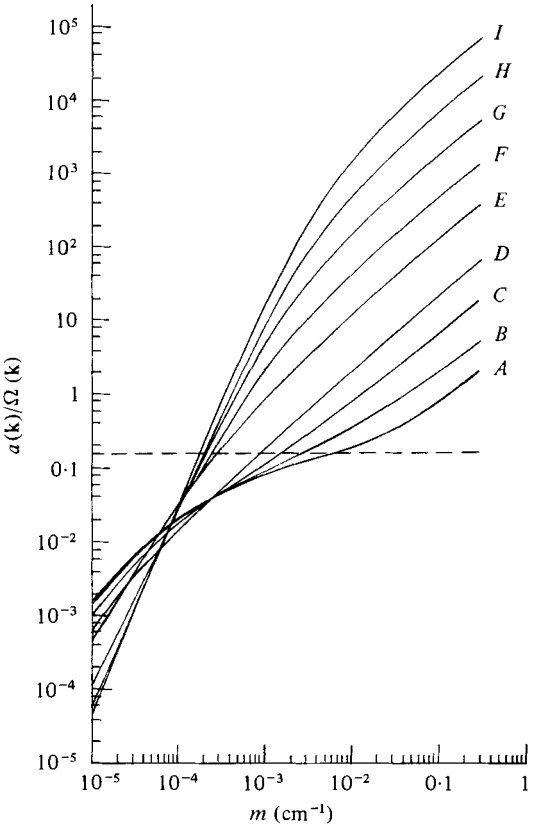

FIGURE 8. Plots of $a(\mathbf{k})/\Omega$ *vs.* m for various frequencies. The values of Ω for each curve are as follows: (*A*) $7{\cdot}0\times10^{-5}$ s^{-1}, (*B*) $7{\cdot}3\times10^{-5}$ s^{-1}, (*C*) $8{\cdot}0\times10^{-5}$ s^{-1}, (*D*) $1{\cdot}1\times10^{-4}$ s^{-1}, (*E*) $1{\cdot}7\times10^{-4}$ s^{-1}, (*F*) $3{\cdot}2\times10^{-4}$ s^{-1}, (*G*) $6{\cdot}2\times10^{-4}$ s^{-1}, (*H*) $1{\cdot}2\times10^{-3}$ s^{-1}, (*I*) $2{\cdot}2\times10^{-3}$ s^{-1}. This labelling applies to all curves in figures 8 and 9. The values of Ω correspond to equal increments in $\log[(k_1^2+k_2^2)/k_3^2]$.

We find the following identifications,

$$\frac{1}{n(\mathbf{k})}\mathscr{S}_2(\mathbf{k}) = \frac{\Sigma_2(\mathbf{k})}{\hat{U}(\mathbf{k},0)} \tag{5.6}$$

and

$$\frac{1}{n(\mathbf{k})}\mathscr{S}_1(\mathbf{k}) = 2(\nu(\mathbf{k})-\nu_0(\mathbf{k})) = 2\mathscr{I}\Sigma_1(\mathbf{k}), \tag{5.7}$$

where $\hat{U}$ is defined in (2.23), Σ_2 is the effective force due to all other modes, and F is the total force,

$$F(\mathbf{k}) = F_0(\mathbf{k})+\Sigma_2(\mathbf{k}). \tag{5.8}$$

From equation (4.4) we have, in the stationary state,

$$F = +2\nu\hat{U}(\mathbf{k},0) = F_0(\mathbf{k})+\Sigma_2(\mathbf{k})$$

or

$$\frac{F_0(\mathbf{k})}{\hat{U}(\mathbf{k},0)} = 2\mathscr{I}\Sigma_1(\mathbf{k}) - \frac{\Sigma_2(\mathbf{k})}{\hat{U}(\mathbf{k},0)} + 2\nu_0(\mathbf{k}), \tag{5.9}$$

$$\frac{F_0(\mathbf{k})}{\hat{U}(\mathbf{k})} - 2\nu_0(\mathbf{k}) = \frac{-\mathscr{S}(\mathbf{k})}{n(\mathbf{k})} = \frac{\mathscr{S}_1(\mathbf{k})}{n(\mathbf{k})} - \frac{\mathscr{S}_2(\mathbf{k})}{n(\mathbf{k})}. \tag{5.10}$$

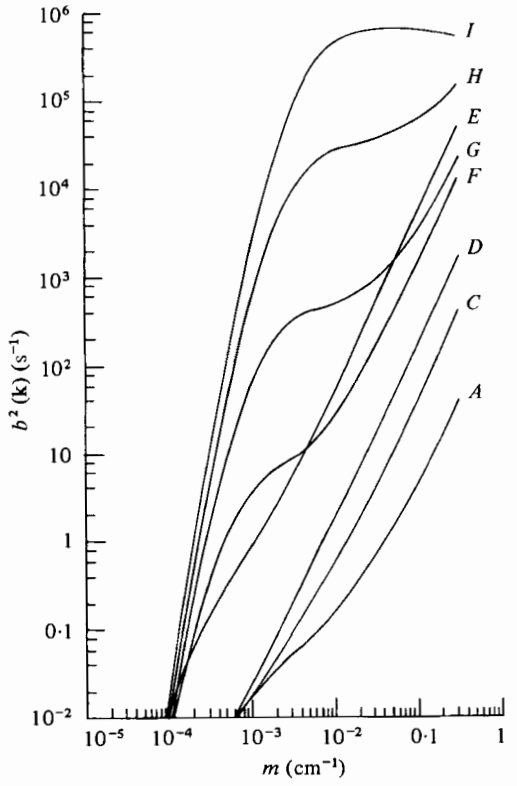

FIGURE 9. Plots of $b^2(\mathbf{k})$ *vs.* m for various frequencies.

From this equation we see that the calculations of Olbers and McComas give the coupling to external systems in terms of the difference between an effective force and an effective viscosity.

Earlier we noted the breakdown of the resonant-interaction approximation (RIA) because of the large values of $\nu(\mathbf{k})$ in some regions of the internal-wave spectrum. We would next like to test the assumptions that Σ_1 and Σ_2 are independent of ω. We consider first the case of Σ_1. In order to ascertain whether the non-resonant structure of $\Sigma_1(\omega)$ might have any appreciable effect on the values computed in the right-hand side of (3.7) we choose a particularly simple form of Σ_1:

$$-i\Sigma_1(\mathbf{k},\omega) = a(\mathbf{k}) + (\omega - \Omega(\mathbf{k}))\, b(\mathbf{k}), \tag{5.11}$$

where both $a(\mathbf{k})$ and $b(\mathbf{k})$ are assumed real. If the Σ_1 = constant assumption is warranted we should expect to compute very small values for $b(\mathbf{k})$. For the present we still assume that Σ_2 = constant and use the RIA. The effect of $b(\mathbf{k})$ is to reduce the residue of the pole of $\mathbf{G}$ from 1 to $1/(1+ib)$ and to decrease the damping from $\nu = a$ to $\nu = a/(1+b^2)$. The correlation function is reduced by the same amount:

$$\hat{U}(\mathbf{k},0) = \int \frac{d\omega}{2\pi}\, U(\mathbf{k},\omega) = \frac{F_0(\mathbf{k}) + \Sigma_2(\mathbf{k})}{2\nu(\mathbf{k})\,(1+b^2(\mathbf{k}))}.$$

We wish to check the size of b numerically. We take $\hat{U}(\mathbf{k},0)$ to be given by GM76, a recently revised version of the Garrett–Murk spectrum (see McComas & Bretherton

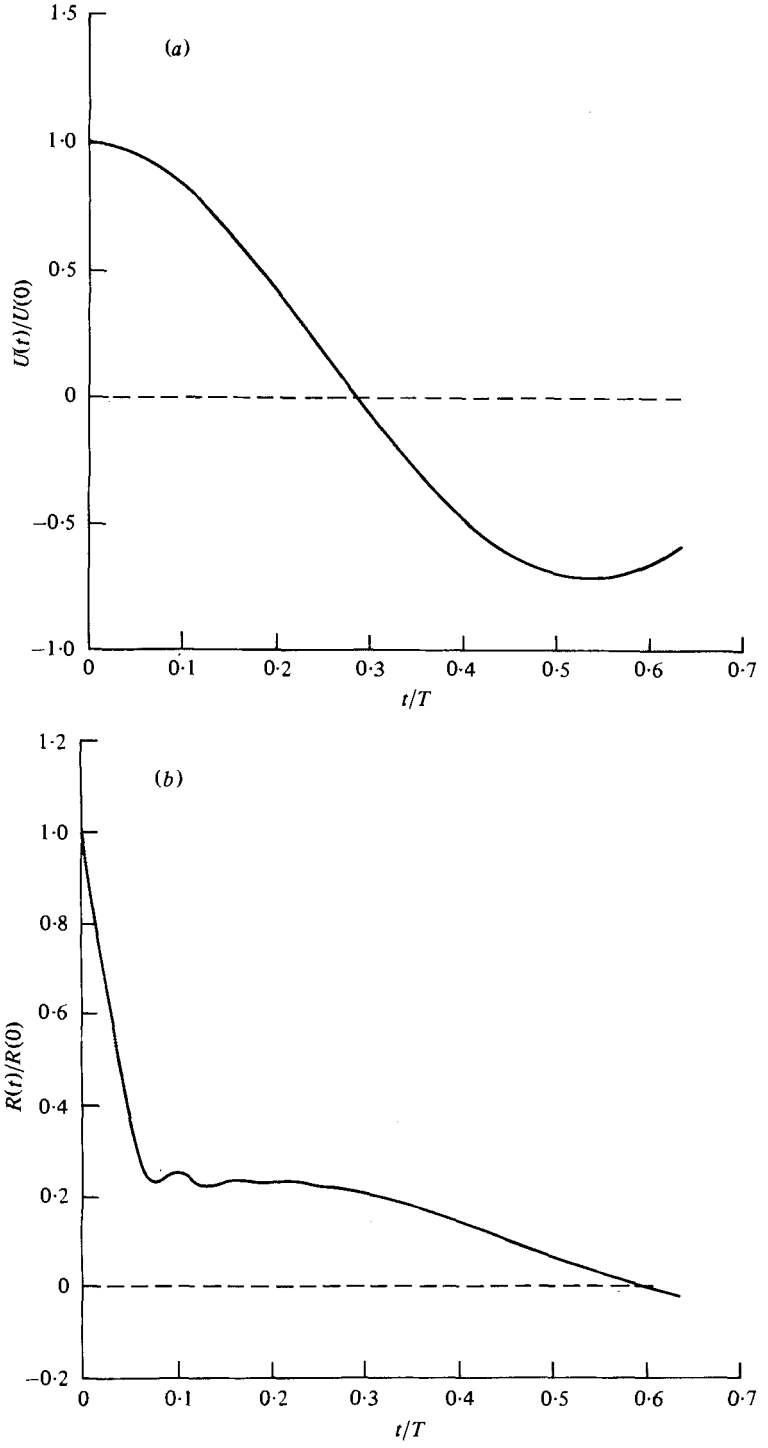

FIGURE 10. (*a*) $U(t)/U(0)$ *vs.* t/T for $\lambda_V = 390$ m and $\Omega/f = 4{\cdot}5$ and (*b*) $R(t)/R(0)$ *vs.* t/T for $\lambda_V = 390$ m and $\Omega/f = 4{\cdot}5$.

1977). The coupling coefficients used in the calculation are derived in the same manner as those given by McComas.

We have computed values of $a(\mathbf{k})$ and $b(\mathbf{k})$ by assuming that both were initially zero (i.e. just $\Sigma_1 \to 0$ as before), and then iterating (3.7) once. Values of $a(\mathbf{k})/\Omega(\mathbf{k})$ and $b^2(\mathbf{k})$ are shown in figures 8 and 9 respectively. We first note the previously mentioned fact that the values of $a(\mathbf{k})$ for much of the spectrum violate

$$a(\mathbf{k})/(\Omega(\mathbf{k})/2\pi) \ll 1.$$

The dashed line corresponds to $a(\mathbf{k})/(\Omega(\mathbf{k})/2\pi) = 1$. The RIA assumes that $\Omega \gg a$; that condition is clearly violated here. We further note that an examination of figure 9 reveals values of $b^2(\mathbf{k})$ that are extremely large for much of the spectrum. This clearly violates the assumption that Σ_1 is constant in ω. We conclude that any reliable self-consistent calculation must include information on the non-resonant structure of Σ_1. It should be noted, however, that the simple linear form of Σ_1, chosen here to test the constant $-\Sigma_1$ assumption, is wholly inappropriate for a serious calculation.

Finally, we can also determine the validity of the assumption that Σ_2 is constant in ω. When we actually compute $\Sigma_2(\mathbf{k}, \omega)$ we find that Σ_2 actually does possess an appreciable structure with a finite width in ω. However it is not sufficient merely to calculate Σ_2 for various values of ω, since frequency filter cuts-off may allow only values of Σ_2 from a narrow band around resonance to enter the calculation. In that case Σ_2 may be considered *effectively* constant. One way to circumvent this is to compare the functions

$$U(t) = \int d\omega\, e^{i\omega t} \frac{\Sigma_2(\omega)}{\nu^2 + (\omega - \Omega)^2}$$

and

$$R(t) = \int d\omega\, e^{i\omega t} \Sigma_2(\omega).$$

We note that Σ_2 always appears in the equations through $U(\omega)$, which contains information on the width of the frequency filter as well as on the width of Σ_2. On the other hand $R(t)$ contains information on the width of Σ_2 only. We envision two possible extreme cases. In one case ν is extremely small, so that the frequency filter becomes a resonance delta function and $U(t)$ assumes the form earlier obtained by assuming Σ_2 = constant. On the other extreme, ν might be so large that the width of the frequency filter would be much larger than the width of Σ_2. In this case we would expect the shapes of $U(t)$ and $R(t)$ to be identical. The calculations that we have performed indicate that both extreme situations do indeed obtain in different regions of the spectrum. Figure 10 shows the first case. Here Σ_2 is constant and $U(t) \propto e^{i\Omega t} e^{-\nu|t|}$. If Σ_2 were actually constant, however, $R(t)$ should be a delta function in t. In actuality we see that $R(t)$ does have some width, indicating that Σ_2 is not constant but that the variation is unimportant in the calculation. On the other hand figure 11 shows the case where, appropriately scaled, $U(t)$ and $R(t)$ are identical, indicating large values of $\nu(\mathbf{k})$. In this region, then, not only is the RIA very bad, but also the assumption that Σ_2 = constant must be dropped if the frequency filter correction is to be used.

We have also calculated $\Sigma_1(\omega)$ and from it the response function to verify directly that the frequency dependence of Σ_1 is important for some modes.

In general, then, we conclude that any self-consistent calculation of Σ_1 and Σ_2 in the region where Olbers and McComas found inconsistencies in their calculations must necessarily include (1) a frequency filter function to relax the RIA, and (2) non-resonant

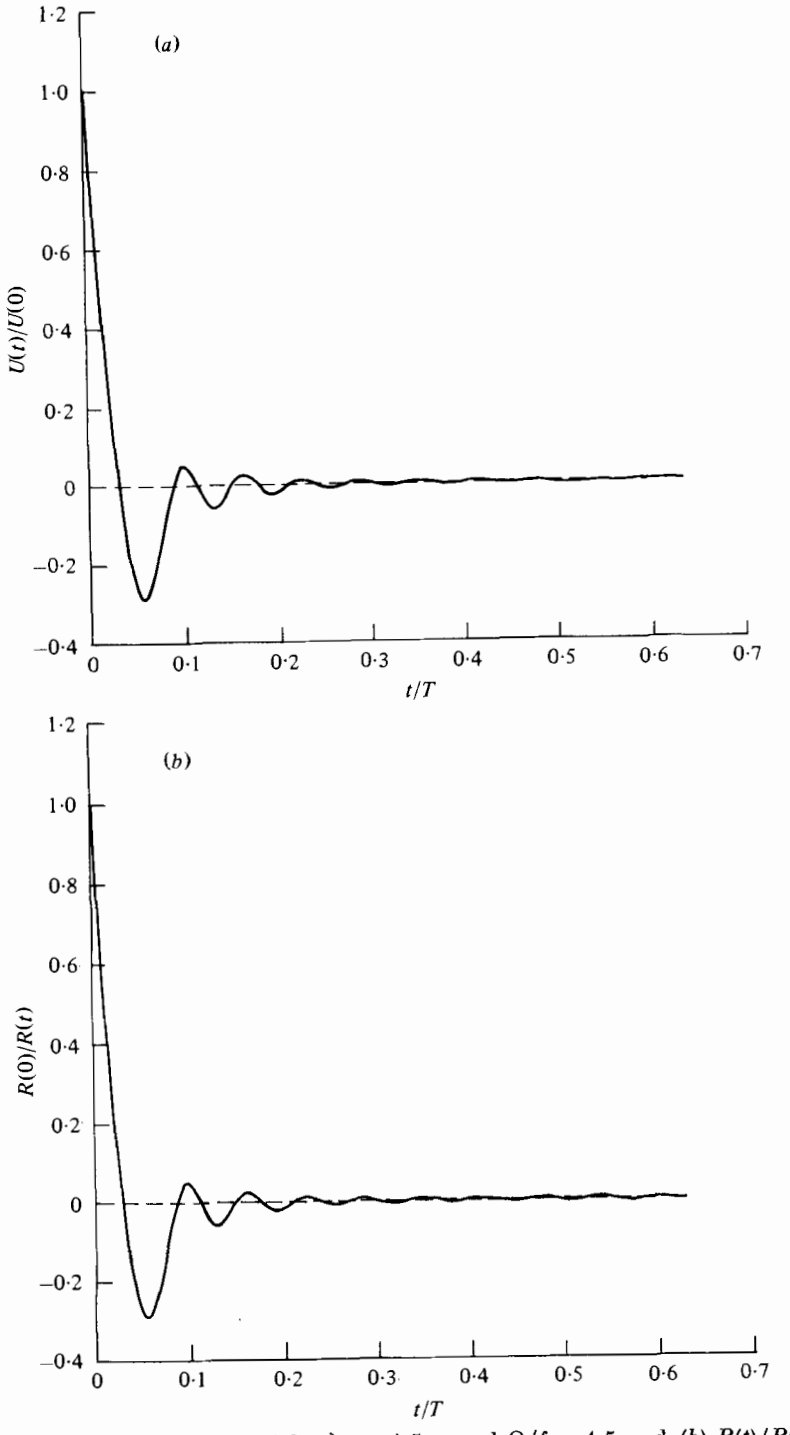

FIGURE 11. (*a*) $U(t)/U(0)$ *vs.* t/T for $\lambda_V = 1{\cdot}5$ m and $\Omega/f = 4{\cdot}5$ and (*b*) $R(t)/R(0)$ *vs.* t/T for $\lambda_V = 1{\cdot}5$ m and $\Omega/f = 4{\cdot}5$.

information on Σ_1 and Σ_2. Unfortunately this greatly increases the difficulty of the numerical solutions of the problem, since it requires an extra integration over ω, including a principal part integral that has not been present in earlier calculations. In addition, the equations must be solved self-consistently, implying an iteration process on an already time-consuming calculation. We are currently attempting a simplified version of the self-consistent calculation.

6. WKB approximation

In the introduction we alluded to other approximations to the path integral. One of the failings of weak interaction theory can be traced to the interaction of two small-scale high-frequency waves with a large-scale inertial frequency wave. This interaction has been called induced diffusion by McComas & Bretherton (1977). It is appropriate to think of it as the effect of advection of the small-scale wave by the inertial wave. This picture suggests studying the propagation of a small-scale wave packet in the geometric optics approximation including the effects of a time-varying background. The aim would be to calculate a coherent decay time. The time scales of the response functions in the induced diffusion region are dominated by processes that leave the wave packet coherent, although somewhat altered.

7. Summary

We have written this paper with a twofold purpose. In the first part of the paper we have introduced a formalism for solving nonlinear equations of the type encountered with internal waves. From experience with self-consistent field treatments in other areas of physics we very well might expect such methods to be helpful in extending calculations beyond some of their inconsistencies. To this end we have taken a simplified steady-state model of the oceanic internal wave field and obtained via path integral methods the corresponding diagrammatic perturbation theory from which Dyson's equations can be obtained. In the process we have suggested as interpretation of the ocean as a collection of uncoupled, damped oscillators driven by random forces and each having a memory. In this case we used the direct interaction approximation of Kraichnan, which is easily implemented in the diagrammatic language. We should note that one problem with this approach is that it is not currently known what kind of approximations guarantee the positivity of quantities like the effective viscosity and also guarantee consistency in the Green's function sum rule. In turbulence problems it has been possible to construct models of fictitious systems that satisfy the DIA exactly, thus guaranteeing the appropriate positivity. However this approach is of very limited usefulness and it is not known how to extend it to other approximations.

In the second part of the paper we have obtained the previous results of Hasselmann via a prescribed limiting process. The particular assumptions inherent in the process were checked and found to break down in the region where Olbers and McComas found inconsistent results in their calculations. We suggested that the calculations might be improved by solving equations for Σ_1 and Σ_2 self-consistently, but that the implementation of the procedure might be quite difficult.

In conclusion, we feel that the formalism of the type discussed here should be

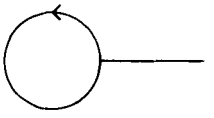

FIGURE 12. Tadpole graph with response function integrated.

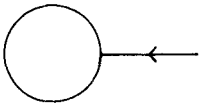

FIGURE 13. Tadpole graph with correlation function integrated.

studied further for its possible application to the internal wave problem. The preliminary calculations indicate ways to correct for deficiencies in earlier calculations, but further work is needed to formulate positively constraints and related constraints.

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Appendix

We discuss briefly the role of the Jacobian (equation (2.14)) in cancelling some tadpole graphs. We will illustrate the cancellation by considering the expectation value of $A(\mathbf{k}, \omega)$, which should be zero. There are two graphs contributing to $\langle A(\mathbf{k}, \omega)\rangle$ and they are shown in figures 12 and 13. If we work to lowest order in B, the Jacobian gives the contribution

$$\langle A^s(\mathbf{k}, \omega)\rangle_{\text{Jacobian}} = 2\int d^3\mathbf{k}_1\, d\omega_1 \sum_{s_1 s_2} \frac{\langle A^{s_1}(\mathbf{k}_1 = 0, \omega_1 = 0)\, A^s(\mathbf{k}, \omega)\rangle_0}{D^{s_1}(\mathbf{k}_1, \omega_1)} B^{s_1 - s_1 s_2}_{k_1 - k_1 0}, \quad \text{(A 1)}$$

where $\langle AA\rangle_0$ is given by (3.1). Using the rules given in the text it is easy to calculate the contribution of figure 12 and verify that it cancels (A 1). The graph shown in figure 13 vanishes for another reason. It is easy to see that it is proportional to

$$\frac{1}{D^s(\mathbf{k} = 0, \omega = 0)} \sum_{s_1} \int d^3\mathbf{k}_1\, d\omega_1\, B^{s s_1 - s}_{0 k_1 - k}\, U^s_0(\mathbf{k}_1, \omega_1)$$

and

$$B^{sxx}_{0xx} \equiv 0.$$

When these graphs are inserted into more complicated graphs they will continue to give no contribution. Similarly more complicated tadpoles (vertex corrections of figures 12 and 13) will continue to vanish.

REFERENCES

EDWARDS, S. F. 1964 The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239–273.

GARRETT, C. J. R. & MUNK, W. H. 1972 Space-time scales of internal waves. *Geophys. Fluid Dyn.* **2**, 225–264.

GARRETT, C. J. R. & MUNK, W. H. 1975 Space-time scales of internal waves: A progress report. *J. Geophys. Res.* **80**, 291–297.

HASSELMANN, K. 1966 Feynman diagrams and interaction rules of wave-wave scattering processes. *Rev. Geophys. Space Phys.* **4**, 1–32.

HASSELMANN, K. 1967 Nonlinear interactions treated by the methods of theoretical physics. *Proc. R. Soc. Lond.* A **291**, 77–100.

HASSELMANN, K. 1968 Weak interaction theory at ocean waves. In *Basic Developments in Fluid Dynamics*, vol. 2, pp. 117–182.

HERRING, J. R. 1965 Self-consistent-field approach to turbulence theory. *Phys. Fluids* **8**, 2219–2225.

HOLLOWAY, G. 1978 Order and disorder in nonlinear fluid motions. *Topics in Nonlinear Dynamics – A Tribute to Sir Edward Bullard.* American Institute of Physics.

HOLLOWAY, G. 1979 On the spectral evolution of strongly interacting waves. *Geophys. Astrophys. Fluid Dyn.* **11**, 271–287.

HOLLOWAY, G. & HENDERSHOTT, M. C. 1977 Stochastic closure for nonlinear Rossby waves. *J. Fluid Mech.* **82**, 747–765.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–542.

LANGOUCHE, F., ROEKAERTS, D. & TIRAPEGUI, E. 1979 Functional integral methods for stochastic fields. *Physica* A **95**, 252.

LESLIE, D. C. 1973 *Developments in the Theory of Turbulence.* Oxford University Press.

MCCOMAS, C. H. 1977 Equilibrium mechanisms within the oceanic internal wave field. *J. Phys. Oceanog.* **7**, 836–845.

MCCOMAS, C. H. & BRETHERTON, F. P. 1977 Resonant interactions of oceanic internal waves. *J. Geophys. Res.* **82**, 1397–1412.

MARTIN, P. C., SIGGIA, E. D. & ROSE, H. A. 1973 Statistical dynamics of classical systems. *Phys. Rev.* A **8**, 423–437.

OLBERS, D. 1976 Nonlinear energy transfer and the energy balance of the internal wave field in the deep ocean. *J. Fluid Mech.* **74**, 375–399.

ORSZAG, S. A. 1977 Lectures on the statistical theory of turbulence. In *Fluid Dynamics* (ed. R. Balian & J. L. Penbe), p. 235. Gordan and Breach.

PHILLIPS, O. M. 1977 *The Dynamics of the Upper Ocean.* Cambridge University Press.

PHYTHIAN, R. 1969 Self-consistent perturbation series for stationary homogeneous turbulence. *J. Phys.* A **2**, 181–192.

PHYTHIAN, R. 1977 The functional formalism of classical statistical dynamics. *J. Phys.* A **10**, 777–789.

POMPHREY, N., MEISS, J. D. & WATSON, K. M. 1980 Description of nonlinear internal wave interactions using Langevin methods. *J. Geophys. Res.* C **85**, 1085–1094.

WYLD, H. W. 1961 Formulation of the theory of turbulence in an incompressible fluid. *Ann. Phys.* **14**, 143–165.